

A regularization approach to non-smooth symplectic geometry

Günther Hörmann ^{*}

Sanja Konjik [†]

Michael Kunzinger [‡]

Abstract

We introduce non-smooth symplectic forms on manifolds and describe corresponding Poisson structures on the algebra of Colombeau generalized functions. This is achieved by establishing an extension of the classical map of smooth functions to Hamiltonian vector fields to the setting of non-smooth geometry. For mildly singular symplectic forms, including the continuous non-differentiable case, we prove the existence of generalized Darboux coordinates in the sense of a local non-smooth pull-back to the canonical symplectic form on the cotangent bundle.

Mathematics Subject Classification (2010): 53D05, 46F30

Keywords: Non-smooth symplectic geometry, Darboux Theorem, Colombeau algebra

1 Introduction

Regularization approaches to non-smooth differential geometry and its applications to mathematical physics have been successfully developed in the context of Colombeau-type generalized functions and tensor fields (cf., e.g., [4, 5, 22, 23, 9]). In the present paper we take up the study of generalized symplectic structures based on previous investigations of linear symplectic structures on modules over Colombeau generalized numbers in [12]. Our main motivations for the systematic development of a non-smooth symplectic differential geometry are driven by deeper applications in microlocal analysis, classical mechanics or general relativity in terms of analysis on semi-Riemannian manifolds:

First, modern research on propagation of singularities for (pseudo-) differential operators with non-smooth principal symbol on a manifold is based on an understanding of the corresponding non-smooth Hamiltonian vector field and its generalized bicharacteristic flow (cf. [6, 24]) as well as on an analysis of microlocal mapping properties of generalized Fourier integral solution operators in terms of the wave front sets of their kernels, which are to be generalized Lagrangian submanifolds (cf. [7, 8]).

Second, non-smooth symplectic structures arise in the study of the geodesic flow in classical mechanics or in general relativity in the context of non-smooth metrics or space-times, unified in

^{*}Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, A-1090 Vienna, Austria, Electronic mail: guenther.hoermann@univie.ac.at

[†]Faculty of Sciences, Department of Mathematics and Informatics, University of Novi Sad, Trg Dositeja Obradovića 4, 21000 Novi Sad, Serbia, Electronic mail: sanja.konjik@dmi.uns.ac.rs

[‡]Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, A-1090 Vienna, Austria, Electronic mail: michael.kunzinger@univie.ac.at.

models of generalized semi-Riemannian manifolds (M, g) . The geodesic flow can then be described in terms of symplectomorphisms on a generalized symplectic manifold (TM, σ) . The basic construction is as follows: The geodesic flow is defined by the non-smooth geodesic spray G , given as a vector field on TM in coordinates (x, v) by

$$G(x, v) = \sum_{1 \leq j \leq n} v_j \partial_{x_j} - \sum_{1 \leq j, k, l \leq n} \Gamma_{kl}^j v_k v_l \partial_{v_j}$$

with the Christoffel symbols Γ_{kl}^j (cf. [1]). Non-degeneracy of the metric provides a ‘non-smooth diffeomorphism’ $g^b: TM \rightarrow T^*M$. The latter allows one to pull-back the canonical symplectic form to define a non-smooth symplectic form σ on TM , which locally reads

$$\sigma = \sum_{1 \leq i, j \leq n} g_{ij} dx_i \wedge dv_j + \sum_{i, j, k} \frac{\partial g_{ij}}{\partial x_k} v_i dx_j \wedge dx_k.$$

Note that in a sense, here the ‘generalized symplectomorphism’ $(g^b)^{-1}$ provides Darboux coordinates for (TM, σ) .

1.1 Generalized differential geometry on smooth manifolds

In this section we briefly recall some notions from Colombeau’s theory of nonlinear generalized functions and non-smooth differential geometry in this setting. For details we refer to [9].

Let M be a smooth (Hausdorff and second countable) manifold of dimension n . Colombeau generalized functions on M are introduced as equivalence classes $u = [(u_\varepsilon)_\varepsilon]$ of moderate modulo negligible nets in $\mathcal{C}^\infty(M)$, where moderateness, resp. negligibility, are characterized by

$$\begin{aligned} \mathcal{E}_M(M) &:= \{(u_\varepsilon)_\varepsilon \in \mathcal{C}^\infty(M)^{[0,1]} : \forall K \subset\subset M, \forall L \in \mathcal{P}(M) \exists N \in \mathbb{N} : \sup_{x \in K} |Lu_\varepsilon(x)| = O(\varepsilon^{-N})\}, \\ \mathcal{N}(M) &:= \{(u_\varepsilon)_\varepsilon \in \mathcal{E}_M(M) : \forall K \subset\subset M, \forall m \in \mathbb{N}_0 : \sup_{x \in K} |u_\varepsilon(x)| = O(\varepsilon^m)\}, \end{aligned}$$

and $\mathcal{P}(M)$ is the space of linear differential operators on M . Then the Colombeau algebra $\mathcal{G}(M)$ of generalized functions on M is defined as is the quotient $\mathcal{E}_M(M)/\mathcal{N}(M)$; it is a fine sheaf of differential algebras with respect to the Lie derivative along smooth vector fields.

Colombeau generalized functions on M are uniquely determined by their values on compactly supported generalized points on M , which are denoted by \widetilde{M}_c and defined as follows. In the space M_c of nets $(x_\varepsilon)_\varepsilon \in M^{[0,1]}$ with the property that x_ε stays in a fixed compact set for ε small, one introduces an equivalence relation \sim : $(x_\varepsilon)_\varepsilon \sim (y_\varepsilon)_\varepsilon \Leftrightarrow d_h(x_\varepsilon, y_\varepsilon) = O(\varepsilon^m)$, for all $m > 0$, $(x_\varepsilon)_\varepsilon, (y_\varepsilon)_\varepsilon \in M_c$, and distance function d_h induced on M by any Riemannian metric h . Then the space of compactly supported generalized points on M is the quotient space $\widetilde{M}_c := M_c / \sim$, with elements $\tilde{x} = [(x_\varepsilon)_\varepsilon]$.

Generalized numbers are equivalence classes $r = [(r_\varepsilon)_\varepsilon]$ of moderate nets of real numbers $\{(r_\varepsilon)_\varepsilon \in \mathbb{R}^{[0,1]} : \exists N \in \mathbb{N} : |r_\varepsilon| = O(\varepsilon^{-N})\}$ modulo the set of negligible nets $\{(r_\varepsilon)_\varepsilon \in \mathbb{R}^{[0,1]} : \forall m \in \mathbb{N}_0 : |r_\varepsilon| = O(\varepsilon^m)\}$. $\widetilde{\mathbb{R}}$ is the ring of constants in the Colombeau algebra of generalized functions on \mathbb{R} . A generalized number $r \in \widetilde{\mathbb{R}}$ is called strictly nonzero if $|r_\varepsilon| \geq \varepsilon^m$ for some $m \in \mathbb{N}$ and ε small; it is called strictly positive if $r_\varepsilon \geq \varepsilon^m$ for some $m \in \mathbb{N}$ and ε small. Invertible elements of the ring $\widetilde{\mathbb{R}}$ are precisely those which are strictly nonzero. Similarly, $u \in \mathcal{G}(M)$ has a multiplicative inverse if and only if it is strictly nonzero in the sense that for any $K \Subset M$ there exists some m such that $\inf_{x \in K} |u_\varepsilon(x)| \geq \varepsilon^m$ for ε small. In turn, this is equivalent to $u(\tilde{x})$ being invertible in $\widetilde{\mathbb{R}}$ for any compactly supported generalized point \tilde{x} .

Let E be a vector bundle over M , and denote by $\Gamma(E)$ the space of smooth sections of E . The space of Colombeau generalized sections of E , $\Gamma_{\mathcal{G}}(E)$, is defined as the quotient $\Gamma_{\mathcal{E}_M}(E)/\Gamma_{\mathcal{N}}(E)$,

where

$$\begin{aligned}\Gamma_{\mathcal{E}_M}(E) &:= \{(s_\varepsilon)_\varepsilon \in \Gamma(E)^{[0,1]} : \forall L \in \mathcal{P}(E) \forall K \subset\subset M \exists N \in \mathbb{N} : \sup_{x \in K} \|Lu_\varepsilon(x)\|_h = O(\varepsilon^{-N})\}, \\ \Gamma_{\mathcal{N}}(E) &:= \{(s_\varepsilon)_\varepsilon \in \Gamma_{\mathcal{E}_M}(E) : \forall K \subset\subset M \forall m \in \mathbb{N} : \sup_{x \in K} \|u_\varepsilon(x)\|_h = O(\varepsilon^m)\},\end{aligned}$$

Here $\mathcal{P}(E)$ is the space of linear differential operators $\Gamma(E) \rightarrow \Gamma(E)$, and $\|\cdot\|_h$ is the norm on the fibers of E induced by any Riemannian metric h on M . The $\mathcal{C}^\infty(M)$ -module of Colombeau generalized sections of E is projective and finitely generated, and can be characterized by the $\mathcal{C}^\infty(M)$ -module isomorphisms: $\Gamma_{\mathcal{G}}(E) = \mathcal{G}(M) \otimes_{\mathcal{C}^\infty(M)} \Gamma(E) = L_{\mathcal{C}^\infty(M)}(\Gamma(E^*), \mathcal{G}(M))$.

In case E is a tensor bundle $T_s^r(M)$ we use the notation $\mathcal{G}_s^r(M)$ for $\Gamma_{\mathcal{G}}(T_s^r(M))$. Moreover, in the case of the tangent bundle TM the generalized sections are the generalized vector fields, and will be denoted by $\mathfrak{X}_{\mathcal{G}}(M)$, while in the case of the cotangent bundle T^*M we write $\Omega_{\mathcal{G}}^1(M)$ for the corresponding generalized sections (generalized one-forms). Also, when E is the vector bundle of exterior k -forms on TM , i.e., $E = \Lambda^k T^*M$, then the generalized sections are the generalized k -forms on M , and are denoted by $\Omega_{\mathcal{G}}^k(M)$.

Finally, we will also make use of a particular feature of Colombeau's approach, namely manifold-valued generalized functions ([14, 16]). Given manifolds M, N , the space of c-bounded generalized functions is the quotient space $\mathcal{G}[M, N] := \mathcal{E}_M[M, N] / \sim$. Here, $\mathcal{E}_M[M, N]$ is the set of all nets $(u_\varepsilon)_{\varepsilon \in]0,1]}$ such that $(f \circ u_\varepsilon)_\varepsilon$ is moderate for every $f \in \mathcal{C}^\infty(N)$. The equivalence relation \sim is defined as $(u_\varepsilon)_\varepsilon \sim (v_\varepsilon)_\varepsilon$ if for any Riemannian metric h on N , any $m \in \mathbb{N}$ and every $K \Subset M$, $\sup_{p \in K} d_h(u_\varepsilon(p), v_\varepsilon(p)) = O(\varepsilon^m)$ for $\varepsilon \rightarrow 0$. These generalized functions are called c-bounded since any representative is bounded, uniformly in ε , on any compact subset of M for ε small. They can be composed unrestrictedly, and invertible c-bounded generalized functions serve as non-smooth analogues of diffeomorphisms in the smooth category (cf. [3, 26]).

The paper is organized as follows. For the sake of completeness we first review in Section 2 some results from [12] on symplectic forms on $\widetilde{\mathbb{R}}$ -modules, symplectic bases, maps, and submodules. Then we turn to the manifold setting, study various conditions for skew-symmetry and nondegeneracy of generalized 2-forms, and provide an equivalent characterization of a generalized symplectic form on a manifold. Section 3 is devoted to the Darboux theorem and its generalization for a generalized symplectic form. We state conditions which imply that a generalized symplectic form on a manifold looks locally like the canonical symplectic form on $T^*(\widetilde{\mathbb{R}}^n)$. In the last Section 4 we introduce notions of generalized Hamiltonian vector fields and Poisson structures.

2 Generalized symplectic structures

2.1 Review of symplectic modules over the ring of generalized numbers

Here we briefly recall basic notions and results about symplectic $\widetilde{\mathbb{R}}$ -modules that are essential for the study of generalized symplectic structures on manifolds, and refer to [12] for an in-depth analysis.

On an $\widetilde{\mathbb{R}}$ -module V we define a symplectic form $\sigma : V \times V \rightarrow \widetilde{\mathbb{R}}$ to be an $\widetilde{\mathbb{R}}$ -bilinear form that is skew-symmetric ($\sigma(v, w) = -\sigma(w, v), \forall v, w \in V$), and non-degenerate ($\sigma(v, w) = 0, \forall w \implies v = 0$). We call the pair (V, σ) a symplectic $\widetilde{\mathbb{R}}$ -module.

As the standard model space for a symplectic $\widetilde{\mathbb{R}}$ -module we take $(T^*(\widetilde{\mathbb{R}}^n), \tilde{\omega})$, where $T^*(\widetilde{\mathbb{R}}^n) = \widetilde{\mathbb{R}}^n \times \widetilde{\mathbb{R}}^n$, and the symplectic form $\tilde{\omega}$ is defined as

$$\tilde{\omega}((x, \xi), (y, \eta)) = \sum_{j=1}^n y_j \xi_j - \sum_{j=1}^n x_j \eta_j = \langle y, \xi \rangle - \langle x, \eta \rangle \quad \forall (x, \xi), (y, \eta) \in T^*(\widetilde{\mathbb{R}}^n). \quad (1)$$

For this symplectic form, the vectors $\{e_1, \dots, e_n, f_1, \dots, f_n\}$, $e_j := (\delta_j, 0)$, $f_j = (0, \delta_j)$ (δ_j is the j^{th} standard unit vector, $1 \leq j \leq n$), form a basis which turns $T^*(\widetilde{\mathbb{R}}^n)$ into a free module of rank $2n$. Moreover, one has

$$\widetilde{\omega}(e_j, e_l) = 0 = \widetilde{\omega}(f_j, f_l), \quad \widetilde{\omega}(f_j, e_l) = \delta_{jl} \quad (1 \leq j, l \leq n),$$

with δ_{jl} the Kronecker delta. A basis satisfying this property is called a symplectic basis.

In [12] we proved that any symplectic free module of finite rank possesses a symplectic basis, which further implies that its rank has to be even. Also, we showed that any given “partial symplectic basis” of the free symplectic $\widetilde{\mathbb{R}}$ -module (V, σ) of finite rank can be extended to a full one, i.e., any free set $B := \{e_i \in V \mid i \in I\} \cup \{f_j \in V \mid j \in J\}$ ($I, J \subseteq \{1, \dots, n\}$) satisfying

$$\sigma(e_i, e_k) = 0 = \sigma(f_j, f_l), \quad \sigma(f_j, e_i) = \delta_{ji} \quad (i, k \in I; j, l \in J),$$

can be extended by vectors $e_i \in V$ ($i \in \{1, \dots, n\} \setminus I$) and $f_j \in V$ ($j \in \{1, \dots, n\} \setminus J$) to a symplectic basis $\{e_1, \dots, e_n, f_1, \dots, f_n\}$ of (V, σ) .

One of the main results of [12] is that any free symplectic $\widetilde{\mathbb{R}}$ -module of finite rank is symplectomorphic to $(T^*(\widetilde{\mathbb{R}}^n), \widetilde{\omega})$. A symplectomorphism between symplectic $\widetilde{\mathbb{R}}$ -modules (V_1, σ_1) and (V_2, σ_2) is an $\widetilde{\mathbb{R}}$ -linear isomorphism $f : V_1 \rightarrow V_2$ that preserves symplectic structures, i.e.,

$$\sigma_2(f(v_1), f(v_2)) = \sigma_1(v_1, v_2), \quad \forall v_1, v_2 \in V.$$

In case f is an $\widetilde{\mathbb{R}}$ -linear map but not an isomorphism, it is called a symplectic map. Every symplectic map is injective. A symplectic map $f : V_1 \rightarrow V_2$ is a symplectomorphism if, in addition, (V_1, σ_1) and (V_2, σ_2) are free and of equal finite rank.

2.2 Manifolds with generalized symplectic forms

The following definition is the natural extension of the notion of a smooth symplectic form on a manifold to the setting of Colombeau generalized functions.

Definition 2.1. *A generalized symplectic form on the smooth d -dimensional manifold M is a closed generalized 2-form $\sigma \in \Gamma_{\mathcal{G}}(\Lambda^2 T^*M)$ that is non-degenerate, i.e., for every chart (W, ψ) and $\tilde{z} \in \psi(W)_c^\sim$ the $\widetilde{\mathbb{R}}$ -bilinear form $\psi_*\sigma(\tilde{z}) : \widetilde{\mathbb{R}}^d \times \widetilde{\mathbb{R}}^d \rightarrow \widetilde{\mathbb{R}}$ is non-degenerate.*

As was already mentioned in the introduction, the fact that $\psi_*\sigma$ is a symplectic form on $\widetilde{\mathbb{R}}^d$ implies that d is even, say $d = 2n$.

Given any $(0, 2)$ -tensor field α on M and a coordinate system $(\psi = (x^1, \dots, x^d), U)$ we may write $\alpha|_U = \alpha_{ij} dx^i \otimes dx^j$ and set $\text{vol}(\alpha)|_U := \sqrt{|\det(\alpha_{ij})|}$. As this quantity transforms by multiplication with the Jacobian determinant of the chart transition functions, we obtain a well-defined 1-density $\text{vol}(\alpha)$ on M . In the case where α is a Riemannian metric on M , $\text{vol}(\alpha)$ is the Riemannian volume density of α . A component-wise application of the above procedure to a generalized $(0, 2)$ -tensor α yields a corresponding generalized one-density $\text{vol}(\alpha) \in \Gamma_{\mathcal{G}}(\text{Vol}(M))$ (with $\text{Vol}(M)$ the 1-density bundle over M).

Proposition 2.2. *Let $\sigma \in \mathcal{G}_2^0(M)$. Then the following are equivalent:*

- (i) $\sigma : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathcal{G}(M)$ is skew-symmetric (equivalently, $\sigma \in \Gamma_{\mathcal{G}}(\Lambda^2 T^*M)$) and $\text{vol}(\sigma)$ is strictly positive.
- (ii) For each chart (W, ψ) and for each $\tilde{z} \in \psi(W)_c^\sim$, the map $\psi_*\sigma(\tilde{z}) : \widetilde{\mathbb{R}}^d \times \widetilde{\mathbb{R}}^d \rightarrow \widetilde{\mathbb{R}}$ is skew-symmetric and non-degenerate.

(iii) $\text{vol}(\sigma)$ is strictly positive and for each relatively compact open set $V \subseteq M$ there exist a representative $(\sigma_\varepsilon)_\varepsilon$ of σ and $\varepsilon_0 > 0$ such that $\sigma_\varepsilon|_V$ is skew-symmetric for all $\varepsilon < \varepsilon_0$.

Proof. (i) \Rightarrow (ii) By the point-value characterization of invertibility of generalized functions ([9, 1.2.55]), strict positivity of $\text{vol}(\sigma)$ implies that $\det(\psi_*\sigma(\tilde{z}))$ is invertible in \mathbb{R} for each $\tilde{z} \in M_c^\sim$. Hence by [9, 1.2.41], $\psi_*\sigma(\tilde{z}) : \widetilde{\mathbb{R}^d} \times \widetilde{\mathbb{R}^d} \rightarrow \mathbb{R}$ is non-degenerate. Skew-symmetry of $\psi_*\sigma(\tilde{z})$ follows by inserting basis vector fields in a chart and evaluating.

(ii) \Rightarrow (i) Given $X_1, X_2 \in \mathfrak{X}(M)$ it follows from (ii) and the point value characterization of Colombeau functions that $\sigma(X_1, X_2)|_W = -\sigma(X_2, X_1)|_W$ on any chart domain W . This gives skew-symmetry since \mathcal{G} is a sheaf. By [9, 1.2.41], for any $\tilde{z} \in \psi(W)_c^\sim$, $\det(\psi_*\sigma(\tilde{z}))$ is invertible, hence $(\psi_*\sigma(\tilde{z}))$ being skew-symmetric) strictly positive, in \mathbb{R} . Therefore $\text{vol}(\sigma)|_W$ is strictly positive. Since W was any chart domain, $\text{vol}(\sigma)$ is strictly positive on M .

(i) \Rightarrow (iii) Using a partition of unity the problem can be reduced to the case of $M = \mathbb{R}^d$. Now pick any representative $(\sigma_\varepsilon)_\varepsilon$ of σ . Then $(\tilde{\sigma}_\varepsilon)_{ij} := \frac{1}{2}((\sigma_\varepsilon)_{ij} - (\sigma_\varepsilon)_{ji})$ gives a skew-symmetric representative of σ (cf. [12, Lemma 3.23]).

(iii) \Rightarrow (i) Let $X_1, X_2 \in \mathfrak{X}(M)$ and let V be any relatively compact open subset of M . Picking a representative as in (iii) it is clear that $\sigma(X_1, X_2) = -\sigma(X_2, X_1)$ on V . Then the sheaf property of \mathcal{G} gives skew-symmetry. \square

To obtain a characterization of symplectic generalized forms from this result, we will use the following generalized Poincaré lemma (see [9, 3.2.40]):

Theorem 2.3. *Let $\alpha \in \Gamma_{\mathcal{G}}(\Lambda^k T^*M)$ be closed. If $p \in M$ and U is a neighborhood of p that is diffeomorphic to an open ball in \mathbb{R}^d then there exists $\beta \in \Gamma_{\mathcal{G}}(\Lambda^{k-1} T^*M)$ such that $\alpha|_U = d\beta|_U$.*

Corollary 2.4. *Let $\sigma \in \mathcal{G}_2^0(M)$. Then the following are equivalent:*

- (i) σ is a generalized symplectic form on M .
- (ii) $\text{vol}(\sigma)$ is strictly positive and for every $p \in M$ there exists an open neighborhood U of p and a representative $(\sigma_\varepsilon)_\varepsilon$ of $\sigma|_U$ such that each σ_ε is a symplectic form on U .

Proof. (ii) \Rightarrow (i) is immediate from Prop. 2.2 (iii).

(i) \Rightarrow (ii) Given $p \in M$, let U be a relatively compact neighborhood of p diffeomorphic (via some chart ψ) to an open ball in \mathbb{R}^d . Then Prop. 2.2 (iii) provides a representative $(\tilde{\sigma}_\varepsilon)_\varepsilon$ of σ as well as some $\varepsilon_0 > 0$ and $m \in \mathbb{N}$ such that $\det(\psi_*\tilde{\sigma}_\varepsilon(x)) > \varepsilon^m$ for $\varepsilon < \varepsilon_0$ and all $x \in \psi(U)$, so $\tilde{\sigma}_\varepsilon|_U$ is a non-degenerate 2-form. Since σ is closed, there exists a negligible net $(n_\varepsilon)_\varepsilon$ of 3-forms such that for all ε we have $d\tilde{\sigma}_\varepsilon = n_\varepsilon$. Thus $dn_\varepsilon = 0$ for all ε . By the classical Poincaré lemma we may write $n_\varepsilon = dm_\varepsilon$ on U , and (the proof of) Th. 2.3 shows that $(m_\varepsilon)_\varepsilon$ is a negligible net of 2-forms on U . Thus $\sigma_\varepsilon := \tilde{\sigma}_\varepsilon - m_\varepsilon$ is a representative of $\sigma|_U$ with $d\sigma_\varepsilon = 0$ for all ε . Finally, non-degeneracy of σ_ε for ε small follows since $\text{vol}(\sigma)$ is strictly positive by Prop. 2.2. \square

Note that σ_ε is non-degenerate if and only if the n -fold exterior product $\sigma_\varepsilon^n := \sigma_\varepsilon \wedge \sigma_\varepsilon \wedge \dots \wedge \sigma_\varepsilon$ provides a volume form on M (with $\dim(M) = d = 2n$).

3 A generalized Darboux theorem for non-smooth symplectic forms

In his fundamental work on the distributional approach to non-smooth mechanics [18], J. E. Marsden states: “*It is meaningful to talk about generalized symplectic forms although this does not lead to a satisfactory theory. Clearly Darboux’s theorem cannot hold in that case.*”

The problem we address in this section is: let σ be a generalized symplectic form on the manifold M (in the sense described in the previous section). Can we find *generalized* Darboux coordinates? That is, for any $p \in M$ we seek an open neighborhood U of p and a generalized diffeomorphism $\Phi \in \mathcal{G}[U, \mathbb{R}^{2n}]$ such that

$$\sigma = \Phi^* \tilde{\omega},$$

where $\tilde{\omega}$ is the canonical 2-form on $T^*(\mathbb{R}^n)$.

Weinstein’s proof (based on an isotopy method by Moser from [21]) outlines the following basic steps to construct Darboux coordinates (cf. [27]; see also [1, Section 3.2], [19, Section 3.2], or [17, Chapter 22]) — we add remarks on the key issues in extending this to the non-smooth situation below:

1. We are dealing with a local question, hence we may assume without loss of generality that $M = \mathbb{R}^{2n}$ and $p = 0$.
2. Define the constant symplectic form β on \mathbb{R}^{2n} by $\beta(x) := \sigma(0)$ for every $x \in \mathbb{R}^{2n}$ and put

$$\mu_t := \sigma + t(\beta - \sigma) \quad (0 \leq t \leq 1).$$

Note that $d\mu_t = 0$, since σ is closed and β is constant; moreover, $\mu_0 = \sigma$ and $\mu_1 = \beta$.

3. On some open ball around 0, say $B_R(0)$, we have that μ_t is non-degenerate for every $t \in [0, 1]$. In the case of generalized 2-forms we will need a condition, called condition (\star) below, ensuring that the same holds uniformly for small values of the parameter ε , if $(\mu_t^\varepsilon)_{\varepsilon \in]0, 1]}$ is a family representing μ_t .
4. Applying Poincaré’s lemma on $B_R(0)$ there is a 1-form α such that $d\alpha = \beta - \sigma$ and $\alpha(0) = 0$.
5. Define a vector field X_t on $B_R(0)$ by requiring $\mu_t(X_t, \cdot) = -\alpha$. Since this corresponds to a pointwise inversion of the matrices representing μ_t , the non-smooth analogue will depend on the precise information from the “non-degeneracy” condition (\star) in a crucial way.
6. Since $X_t(0) = 0$ we have an evolution $\theta_{t,s}$ for the time-dependent vector field X_t defined up to $t = 1$ on some possibly smaller neighborhood U of 0 (note that $\theta_{t,0}(0) = 0$ for all t and $\theta_{0,0}$ is the identity). In the non-smooth case, again exploiting details from condition (\star) will be essential to establish a generalized evolution correspondingly.
7. The result of the following calculation (using [17, Proposition 22.15] to obtain the first equality)

$$\begin{aligned} \frac{d}{dt}(\theta_{t,0}^* \mu_t) &= \theta_{t,0}^*(L_{X_t} \mu_t) + \theta_{t,0}^* \frac{d}{dt} \mu_t \\ &= \theta_{t,0}^*(d i_{X_t} \mu_t + i_{X_t} d\mu_t) + \theta_{t,0}^*(\beta - \sigma) = \theta_{t,0}^*(-d\alpha + \beta - \sigma) = 0 \end{aligned}$$

implies $\theta_{1,0}^* \beta = \theta_{1,0}^* \mu_1 = \theta_{0,0}^* \mu_0 = \sigma$, hence in coordinates corresponding to the diffeomorphism $\theta_{1,0}$ the symplectic form σ is the constant form β .

8. Finally, we may map β into the canonical 2-form (or $\tilde{\omega}$ in the generalized setting) by the Darboux-analogue of symplectic linear algebra ([12, Th. 3.3]), which in combination with the previous step yields the desired transformation of σ .

Turning now to the generalized setting, let σ be a generalized symplectic form on the smooth $2n$ -dimensional manifold M . The basic condition on σ , which guarantees the existence of generalized Darboux coordinates, is that σ should possess a representative $(\sigma_\varepsilon)_\varepsilon$ satisfying:

- (\star) The family $(\sigma_\varepsilon)_{\varepsilon \in]0,1]}$ of maps $M \rightarrow \Lambda^2 T^*M$ is equicontinuous and satisfies the following on any chart (W, ψ) with domain $W \subseteq M$ and matrix representation $\Omega^\varepsilon: W \rightarrow M(2n, \mathbb{R})$ of σ_ε with respect to this chart: $\forall K \Subset W \exists C_1, C_2 > 0 \exists \eta > 0 \forall \varepsilon \in]0, \eta] \forall q \in K$

$$0 < C_1 \leq \min \mathcal{A}(\Omega^\varepsilon(q)) \leq \max \mathcal{A}(\Omega^\varepsilon(q)) \leq C_2.$$

Here, for a matrix $B \in M(2n, \mathbb{R})$ we let $\mathcal{A}(B) := \{|\lambda| \mid \lambda \text{ is an eigenvalue of } B\}$.

Remark 3.1.

- (i) Condition (\star) holds for the typical convolution-type regularization of a uniformly continuous symplectic form.
- (ii) (\star) requires that the ε -parametrized family of volume forms $\sigma_\varepsilon^n := \sigma_\varepsilon \wedge \sigma_\varepsilon \wedge \dots \wedge \sigma_\varepsilon$ has uniform bounds on how a $2n$ -dimensional volume is squeezed or stretched.
- (iii) If σ satisfies (\star) and, in addition, is associated to some $\sigma_0 \in \Omega_{\mathcal{D}}^2(M)$, then by Arzela-Ascoli there exists some sub-sequence $(\sigma_{\varepsilon_k})_{k \in \mathbb{N}}$ that converges locally uniformly to a continuous two-form σ_1 . Since $\sigma_\varepsilon \rightarrow \sigma_0$ in distributions it follows that $\sigma_0 = \sigma_1$ is continuous.
- (iv) The generalized 2-forms $\varepsilon dx \wedge d\xi$ and $\frac{1}{\varepsilon} dx \wedge d\xi$ are simple examples of generalized symplectic forms that do not satisfy (\star), yet can be transformed to the canonical symplectic form. However, no such transformation can be c-bounded.
- (v) Writing H and y_+ for embeddings of the Heaviside function resp. its primitive into the Colombeau algebra, consider $\sigma := (1 + H(x))dx \wedge d\xi$. Then (by (iii)), σ does not satisfy (\star). Nevertheless, setting $\Psi(y, \eta) := (y - \frac{1}{2}y_+, \eta)$ we obtain $\Psi_*\sigma = dx \wedge d\xi$. Moreover, Ψ is a generalized diffeomorphism, and both Ψ and Ψ^{-1} are associated to Lipschitz continuous transformations.

Theorem 3.2. *Let σ be a generalized symplectic form on the $2n$ -dimensional smooth manifold M with representative $(\sigma_\varepsilon)_{\varepsilon \in]0,1]}$ such that (\star) holds. Then every $p \in M$ possesses a neighborhood U and a generalized diffeomorphism $\Phi \in \mathcal{G}[U, V]$ with V open in $T^*(\mathbb{R}^n)$ such that $\Phi_*(\sigma|_U) = \tilde{\omega}$.*

Proof. *Step 1:* This can be carried out as in the above scheme: we may assume that $M = \mathbb{R}^{2n}$ and that $p = 0$. Then each σ_ε is a skew-symmetric and non-degenerate matrix. In particular, $\Omega^\varepsilon = \sigma_\varepsilon$ in (\star).

Step 2: Setting $\beta := \sigma(0)$, β is a symplectic form on the $\tilde{\mathbb{R}}$ -module $T^*(\tilde{\mathbb{R}}^n) = \tilde{\mathbb{R}}^n \times \tilde{\mathbb{R}}^n$ and clearly $d\beta = 0$. For $t \in [0, 1]$, set $\mu_t^\varepsilon := \sigma_\varepsilon + t(\beta_\varepsilon - \sigma_\varepsilon)$. Then $\mu_0 = \sigma$ and $\mu_1 = \beta$ in $\Gamma_{\mathcal{G}}(\Lambda^2 T^*\mathbb{R}^n) = \Omega_{\mathcal{G}}(\mathbb{R}^{2n})$, and $d\mu_t = 0$.

Step 3: By assumption (\star), the family of matrix-valued maps

$$\mu^\varepsilon := (t, q) \mapsto \mu_t^\varepsilon(q), \quad [0, 1] \times \mathbb{R}^{2n} \rightarrow M(2n, \mathbb{R}) \quad (\varepsilon \in]0, 1])$$

is equicontinuous. We show that for small $\varepsilon > 0$, μ^ε maps $[0, 1]$ times some fixed neighborhood of 0 into the set of invertible matrices (corresponding to the non-degeneracy of the associated 2-forms) with uniform bounds on the operator norms of the matrices and their inverses. The precise claim is as follows:

($\star\star$) $\exists R > 0 \exists \varepsilon_0 > 0 \exists D > 0$ such that $\forall \varepsilon \in]0, \varepsilon_0] \forall t \in [0, 1] \forall q \in B_R(0)$ the matrix $\mu_t^\varepsilon(q)$ is invertible and

$$\|\mu_t^\varepsilon(q)\|_{\text{op}} \leq D, \quad \|\mu_t^\varepsilon(q)^{-1}\|_{\text{op}} \leq D.$$

To see this, note first that uniform boundedness of $\mu_t^\varepsilon(q)$ for $q \in B_1(0)$ (or any relatively compact subset of \mathbb{R}^{2n}) and small ε follows from

$$\|\mu_t^\varepsilon(q)\|_{\text{op}} \leq (1-t)\|\sigma_\varepsilon(q)\|_{\text{op}} + t\|\sigma_\varepsilon(0)\|_{\text{op}} \leq \|\sigma_\varepsilon(q)\|_{\text{op}} + \|\sigma_\varepsilon(0)\|_{\text{op}},$$

recalling that the operator norm equals the spectral radius for (skew-symmetric hence) normal operators, and finally applying (\star) with $K = \overline{B_1(0)}$. Hence there is some $\eta \in]0, 1]$ and $C_2 > 0$ such that

$$\|\mu_t^\varepsilon(q)\|_{\text{op}} \leq 2C_2 \quad \forall \varepsilon \in]0, \eta] \forall t \in [0, 1] \forall q \in B_1(0).$$

We will also establish invertibility and uniform boundedness of the family of inverses on some ball $B_R(0)$ with $0 < R \leq 1$ and for $0 < \varepsilon \leq \eta$. To this end, we note that $-\sigma_\varepsilon(0)^2$ is a (self-adjoint) positive-definite operator, hence

$$\inf_{\|v\|=1} \|\sigma_\varepsilon(0)v\|^2 = \inf_{\|v\|=1} v^T \cdot (-\sigma_\varepsilon(0)^2) \cdot v = \min \mathcal{A}(-\sigma_\varepsilon(0)^2) = \left(\min \mathcal{A}(\sigma_\varepsilon(0)) \right)^2.$$

Let C_1 denote the (positive) lower bound in (\star) applied as above with $K = \overline{B_1(0)}$, then the previous observation gives $\|\sigma_\varepsilon(0)v\| \geq \min \mathcal{A}(\sigma_\varepsilon(0))\|v\| \geq C_1\|v\|$ and therefore

$$\|\sigma_\varepsilon(0)^{-1}\|_{\text{op}} \leq \frac{1}{C_1} \quad \forall \varepsilon \in]0, \eta].$$

In completing the proof of ($\star\star$) we will make use of the following well-known fact about invertibility in normed algebras with unit (e.g., [13, p. 177]): If A is invertible and $\|B - A\| < 1/\|A^{-1}\|$, then B is invertible and $\|B^{-1} - A^{-1}\| \leq \|B - A\| \|A^{-1}\|^2 / (1 - \|A^{-1}\| \|B - A\|)$. We will apply this to the situation with $A = \sigma_\varepsilon(0)$ and $B = \mu_t^\varepsilon(q)$.

Since we have shown $C_1 \leq 1/\|\sigma_\varepsilon(0)^{-1}\|_{\text{op}}$ above, proving a uniform estimate of the form

$$\|\mu_t^\varepsilon(q) - \sigma_\varepsilon(0)\|_{\text{op}} \leq C_1/2 \quad \forall \varepsilon \in]0, \eta] \forall t \in [0, 1] \forall q \in B_R(0) \quad (\Delta)$$

will suffice to establish invertibility of $\mu_t^\varepsilon(q)$ and a uniform bound

$$\|\mu_t^\varepsilon(q)^{-1}\|_{\text{op}} \leq \|\mu_t^\varepsilon(q)^{-1} - \sigma_\varepsilon(0)^{-1}\|_{\text{op}} + \|\sigma_\varepsilon(0)^{-1}\|_{\text{op}} \leq \frac{\frac{C_1}{2} \frac{1}{C_1^2}}{1 - \frac{1}{C_1} \frac{C_1}{2}} + \frac{1}{C_1} = \frac{2}{C_1}.$$

To argue that (Δ) holds for some $0 < R \leq 1$ we simply call on the equicontinuity of $(\sigma_\varepsilon)_{\varepsilon \in]0, 1]}$ to establish the last inequality in the following chain:

$$\|\mu_t^\varepsilon(q) - \sigma_\varepsilon(0)\|_{\text{op}} = \|(1-t)(\sigma_\varepsilon(q) - \sigma_\varepsilon(0))\|_{\text{op}} \leq \|\sigma_\varepsilon(q) - \sigma_\varepsilon(0)\|_{\text{op}} \leq \frac{C_1}{2}.$$

Therefore ($\star\star$) holds with $D := \max(2C_2, 2/C_1)$.

Step 4: By Th. 2.3 we may construct a generalized 1-form α on $B_R(0)$ such that $d\alpha = \beta - \sigma$. Moreover, the proof of [9, 3.2.40] shows that α has a representative given by

$$\alpha_\varepsilon(q)(v) = \int_0^1 t(\beta_\varepsilon - \sigma_\varepsilon)(tq)(q, v) dt \quad (v \in \mathbb{R}^{2n}) \quad (2)$$

In particular, $\alpha_\varepsilon(0) = 0$ for all ε .

Step 5: By non-degeneracy of μ_t^ε , for each ε there exists a unique vector field X_t^ε on $B_R(0)$ such that $\mu_t^\varepsilon(X_t^\varepsilon, \cdot) = -\alpha_\varepsilon$ on $B_R(0)$ for all $t \in [0, 1]$ and all ε . Moderateness of α_ε and μ_t^ε , together with boundedness of $(\mu_t^\varepsilon)^{-1}$ imply that the net $(X_t^\varepsilon)_\varepsilon$ defines a generalized time-dependent vector field on $B_R(0)$, satisfying global bounds with respect to $t \in [0, 1]$. Next, let $R' < R$ and pick a smooth, compactly supported plateau function $\varphi : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ such that $\varphi \equiv 1$ on a neighborhood of $B_{R'}(0)$. Then $Y_t^\varepsilon(q) := \varphi(q)X_t^\varepsilon(q)$ defines a global time-dependent generalized vector field Y_t that coincides with X_t on $B_{R'}(0)$.

Step 6: Denote by θ^ε the flow of Y_t^ε , i.e.,

$$\begin{aligned} \frac{d}{dt}\theta^\varepsilon(t, s, q) &= Y_t^\varepsilon(\theta^\varepsilon(t, s, q)) \\ \theta^\varepsilon(s, s, q) &= q \end{aligned}$$

Since by (\star) and the above construction Y_t^ε is globally bounded, uniformly in t and ε , each θ^ε is defined on all of $[0, 1] \times [0, 1] \times \mathbb{R}^{2n}$. Thus we obtain a c-bounded generalized function $\theta = [(\theta^\varepsilon)_\varepsilon] \in \mathcal{G}[[0, 1] \times [0, 1] \times \mathbb{R}^{2n}, \mathbb{R}^{2n}]$, and for each fixed $(t, s) \in [0, 1]^2$ an invertible generalized map $\theta_{t,s} := \theta(t, s, \cdot)$.

By condition (\star) it follows that the family X_t^ε ($\varepsilon \in]0, 1]$) is equicontinuous, uniformly in t . The same therefore is true of Y_t^ε ($\varepsilon \in]0, 1]$). Together with the global boundedness of Y_t^ε and the continuous dependence of θ^ε on the right hand side of its defining equation this implies that also θ^ε ($\varepsilon \in]0, 1]$) is equicontinuous. Since $Y_t^\varepsilon(0) = 0$, and thereby $\theta^\varepsilon(t, 0, 0) = 0$ for all t , it follows that there exists some $R'' < R'$ such that $\theta^\varepsilon(t, 0, q) \in B_{R''}(0)$ for all ε , all $q \in B_{R''}(0)$ and all $t \in [0, 1]$.

Step 7: By the above preparations, on $B_{R''}(0)$ we can calculate as follows:

$$\begin{aligned} \frac{d}{dt}((\theta_{t,0}^\varepsilon)^* \mu_t^\varepsilon) &= (\theta_{t,0}^\varepsilon)^*(L_{Y_t^\varepsilon} \mu_t^\varepsilon) + (\theta_{t,0}^\varepsilon)^* \frac{d}{dt} \mu_t^\varepsilon = (\theta_{t,0}^\varepsilon)^*(L_{X_t^\varepsilon} \mu_t^\varepsilon) + (\theta_{t,0}^\varepsilon)^* \frac{d}{dt} \mu_t^\varepsilon \\ &= (\theta_{t,0}^\varepsilon)^*(d i_{X_t^\varepsilon} \mu_t^\varepsilon + i_{X_t^\varepsilon} d \mu_t^\varepsilon) + (\theta_{t,0}^\varepsilon)^*(\beta_\varepsilon - \sigma_\varepsilon) = (\theta_{t,0}^\varepsilon)^*(-d\alpha_\varepsilon + \beta_\varepsilon - \sigma_\varepsilon). \end{aligned}$$

Integrating, we obtain

$$(\theta_{1,0}^\varepsilon)^* \beta_\varepsilon = (\theta_{1,0}^\varepsilon)^* \mu_1^\varepsilon = (\theta_{0,0}^\varepsilon)^* \mu_0^\varepsilon + \nu_\varepsilon = \sigma_\varepsilon + \nu_\varepsilon$$

with $(\nu_\varepsilon)_\varepsilon$ a negligible 2-form. Thus $\sigma = (\theta_{1,0}^\varepsilon)^* \beta$ as a generalized 2-form.

Consequently, the generalized diffeomorphism $\theta_{1,0}$ transforms σ into the constant symplectic form β .

Step 8: Finally, by choosing (according to [12, Th. 3.3]) a symplectic basis on $\widetilde{\mathbb{R}}^{2n}$ corresponding to β we transform β to the canonical symplectic form $\widetilde{\omega}$. \square

4 Generalized Hamiltonian vector fields and Poisson structures

The natural next step in the development of generalized symplectic geometry is the introduction of Hamiltonian vector fields and, building on this, Poisson structures. This final section is devoted to providing these notions.

To begin with, we analyze the purely algebraic setup, based on [12]. Given an $\widetilde{\mathbb{R}}$ -module V , its dual $\widetilde{\mathbb{R}}$ -module $L(V, \widetilde{\mathbb{R}})$ is denoted by V' .

Lemma 4.1. *Let (V, σ) be a symplectic $\widetilde{\mathbb{R}}$ -module that is free and of finite rank. Then for any $\varphi \in V'$ there exists a unique $h_\varphi \in V$, the Hamiltonian vector corresponding to φ , such that*

$$\forall v \in V : \quad \varphi(v) = \sigma(h_\varphi, v).$$

Furthermore, the map $\varphi \mapsto h_\varphi$, $V' \rightarrow V$ is a linear isomorphism

Proof. Uniqueness is immediate by the non-degeneracy of σ . To prove existence, picking any basis of V and the corresponding dual basis on V' we can rewrite the defining equation for h_φ in matrix form as $\varphi \cdot v = h_\varphi^T \cdot \sigma \cdot v$. Hence

$$h_\varphi^T = \varphi \cdot \sigma^{-1}. \quad (3)$$

More explicitly, in terms of a symplectic basis $(e_1, \dots, e_n, f_1, \dots, f_n)$ of V , we have

$$h_\varphi = \sum_{i=1}^n (\varphi(f_i)e_i - \varphi(e_i)f_i).$$

The final claim is immediate from the construction. \square

Based on this result, we can introduce the Poisson bracket

$$\begin{aligned} \{ , \} : V' \times V' &\rightarrow \widetilde{\mathbb{R}} \\ \{\varphi, \psi\} &= \sigma(h_\varphi, h_\psi) \end{aligned}$$

As in the vector space setting it is easily seen that $\{ , \}$ is skew-symmetric, non-degenerate and satisfies the Jacobi-identity. Thus we obtain:

Proposition 4.2. *Let (V, σ) be a symplectic $\widetilde{\mathbb{R}}$ -module that is free and of finite rank. Then $(V', \{ , \})$ is a symplectic $\widetilde{\mathbb{R}}$ -module that is free and of the same rank as V .*

Turning now to the manifold setting, we have:

Theorem 4.3. *Let (M, σ) be a generalized symplectic manifold. Then the mapping*

$$\begin{aligned} \sigma_b : \mathfrak{X}_{\mathcal{G}}(M) &\rightarrow \Omega_{\mathcal{G}}^1(M) \\ \sigma_b(X)(Y) &:= \sigma(X, Y) \quad (X, Y \in \mathfrak{X}_{\mathcal{G}}(M)) \end{aligned}$$

is a $\mathcal{G}(M)$ -linear isomorphism. Its inverse will be denoted by σ^\sharp .

Proof. Since $Y \mapsto \sigma(X, Y)$ is $\mathcal{G}(M)$ -linear, $\sigma_b(X) \in \Omega_{\mathcal{G}}^1(M)$ for any $X \in \mathfrak{X}_{\mathcal{G}}(M)$ (see [9, 3.2.27]), and clearly σ_b is $\mathcal{G}(M)$ -linear. To prove injectivity, suppose that $\sigma(X, Y) = 0$ for all $Y \in \mathfrak{X}_{\mathcal{G}}(M)$. Then for any chart (W, ψ) it follows that $\psi_*(\sigma)(\tilde{z})(\psi_*(X)(\tilde{z}), w) = 0$ for any $\tilde{z} \in \psi(W)_c^\sim$ and any $w \in \widetilde{\mathbb{R}}^{2n}$. Thus non-degeneracy gives $\psi_*(X)(\tilde{z}) = 0$, implying that $X = 0$.

To show surjectivity, by the sheaf property it suffices to consider the case $M = \mathbb{R}^{2n}$. Let $\alpha \in \Omega_{\mathcal{G}}^1(\mathbb{R}^{2n})$. Then, using (3), in matrix notation we may set $X_\alpha^T := \alpha \cdot \sigma^{-1}$. By the positivity of $\det(\sigma)$ in $\mathcal{G}(\mathbb{R}^{2n})$ (Cor. 2.4), this defines a generalized vector field $X_\alpha \in \mathfrak{X}_{\mathcal{G}}(M)$, and by construction $\sigma_b(X_\alpha) = \alpha$. \square

Remark 4.4. Note that the previous result did not make use of generalized Darboux coordinates (which in general may not be available), but is valid for arbitrary generalized symplectic manifolds.

Based on this result, we may now define:

Definition 4.5. *Let (M, σ) be a generalized symplectic manifold. For any $f \in \mathcal{G}(M)$, the generalized vector field $H_f := \sigma^\sharp(df) \in \mathfrak{X}_{\mathcal{G}}(M)$ is called the Hamiltonian vector field of f .*

Moreover, for $f, g \in \mathcal{G}(M)$, the Poisson bracket of f and g is given by $\{f, g\} := \sigma(H_f, H_g)$.

Analogous to the smooth setting, the Poisson bracket induces an $\widetilde{\mathbb{R}}$ -Lie-algebra structure on $\mathcal{G}(M)$.

Summing up, the above constructions provide the foundations for a regularization-based approach to non-smooth symplectic geometry. Contrary to the distributional setting, the additional flexibility of Colombeau’s theory allows one to retain the basic structure of the smooth setting. Building on these constructions one may now systematically explore applications, in particular in non-smooth mechanics. In particular, previous work in this direction (e.g., [10, 15]) can now be viewed from a unifying perspective. Finally, we hope that this approach will be useful in studying the propagation of singularities for pseudo-differential operators with non-smooth principal symbol on differentiable manifolds.

Acknowledgment

This work was supported by project P25236 of the Austrian Science Fund and projects 174024 of the Serbian Ministry of Science, and 114-451-3605 of the Provincial Secretariat for Science.

References

- [1] Abraham, R., Marsden, J. E. *Foundations of Mechanics*. Perseus Books, Cambridge, Massachusetts, second edition, 1978.
- [2] Bartscher, A., Kunzinger, M. Algebras of generalized functions with smooth parameter dependence. *Proc. Edinb. Math. Soc.* (2) 55, no. 1, 105–124, 2012.
- [3] Erlacher, E., Grosser, M. Inversion of a ‘discontinuous coordinate transformation’ in general relativity. *Applicable Analysis* 90: 1707–1728, 2011.
- [4] Colombeau, J.-F. *New generalized functions and multiplication of distributions*. North-Holland, Amsterdam, 1984.
- [5] Colombeau, J.-F., *Elementary introduction to new generalized functions*. North-Holland, Amsterdam, 1985.
- [6] Garetto, C., Hörmann, G. Microlocal analysis of generalized functions: pseudodifferential techniques and propagation of singularities. *Proc. Edinb. Math. Soc.* (2) 48, no. 3, 603–629, 2005.
- [7] Garetto, C., Hörmann, G., Oberguggenberger, M. Generalized oscillatory integrals and Fourier integral operators. *Proc. Edinb. Math. Soc.* (2) 52, no. 2, 351–386, 2009.
- [8] Garetto, C., Oberguggenberger, M. Generalised Fourier integral operator methods for hyperbolic equations with singularities, *Proc. Edinburgh Math. Soc.* 57, 423–463, 2014.
- [9] Grosser, M., Kunzinger, M., Oberguggenberger, M., Steinbauer, R. *Geometric Theory of Generalized Functions*, volume 537 of *Mathematics and its Applications*. Kluwer Academic Publishers, Dordrecht, 2001.
- [10] Hermann, R., Oberguggenberger, M. Ordinary differential equations and generalized functions. *Nonlinear theory of generalized functions*, 85–98, Chapman & Hall/CRC, 1999.
- [11] Hörmander, L. *The analysis of linear partial differential operators III*. Grundlehren der Mathematischen Wissenschaften Vol. 274 Springer-Verlag, Berlin, 1985.
- [12] Hörmann, G., Konjik, S., Kunzinger, M. Symplectic modules over Colombeau-generalized numbers. *Comm. Algebra*, to appear, 2014.

- [13] Kadison, R. V. Ringrose, J. R. Fundamentals of the theory of operator algebras. Vol. I. Elementary theory. Graduate Studies in Mathematics, 15. American Mathematical Society, Providence, RI, 1997.
- [14] Kunzinger, M. Generalized functions valued in a smooth manifold. *Monatsh. Math.*, 137, 31-49, 2002.
- [15] Konjik, S., Kunzinger, M., Oberguggenberger, M., Foundations of the calculus of variations in generalized function algebras. *Acta Appl. Math.* 103, no. 2, 169–199, 2008.
- [16] Kunzinger, M., Steinbauer, R., Vickers, J. A., Intrinsic characterization of manifold-valued generalized functions. *Proc. London Math. Soc.*, 87 (2), 451-470, 2003.
- [17] Lee, J. L. *Introduction to Smooth Manifolds*. Graduate Texts in Mathematics Vol. 218 Springer Science+Business Media, New York, second edition 2013.
- [18] Marsden, J. E. Generalized Hamiltonian mechanics. *Arch. Rat. Mech. Anal.*, 28(4):323–361, 1968.
- [19] McDuff, D., Salamon, D. *Introduction to symplectic topology*. Oxford University Press, New York, second edition 1998.
- [20] Mayerhofer, E. On Lorentz geometry in algebras of generalized functions. *Proc. R. Soc. Edinb. Sect. A, Math.*, 138(4):843–871, 2008.
- [21] Moser, J. K. On the volume elements on manifolds. *Trans. Amer. Math. Soc.*, 120:280–296, 1965.
- [22] Nedeljkov, M., Pilipović, S., Scarpalézos, D. *The linear theory of Colombeau generalized functions*. Pitman Research Notes in Mathematics Series, 385. Longman, Harlow, 1998.
- [23] Oberguggenberger, M. *Multiplication of distributions and applications to partial differential equations*. Pitman Research Notes in Mathematics 259. Longman, 1992.
- [24] Oberguggenberger, M. Hyperbolic systems with discontinuous coefficients: generalized wave-front sets. In *New developments in pseudo-differential operators*, Oper. Theory Adv. Appl., Volume 189, 117–136. Birkhäuser. Basel, 2009
- [25] Oberguggenberger, M., Pilipović, S., Scarpalézos, D. Positivity and positive definiteness in generalized function algebras. *J. Math. Anal. Appl.* 328, no. 2, 1321–1335, 2007.
- [26] Vernaëve, H. Isomorphisms of algebras of generalized functions, *Monatsh. Math.* 162: 225-237, 2011.
- [27] Weinstein, A. Symplectic manifolds and their Lagrangian submanifolds. *Adv. Math.* 6:329–346, 1971.